[13.7] SO(3) is the group of rotations of the unit sphere in 3-space. O(3) extends SO(3) by including reflections. **(A)** Show that SO(3) is a normal subgroup of O(3) and **(B)** show that it is the only proper normal subgroup.

**Note.** (B) is actually not true. There is one other proper normal subgroup of O(3). If 1 is the identity element (null rotation) of SO(3) and R is the reflection operator then **I** ={ 1, R(1) } is a normal subgroup of O(3). This is because if *g* is a (reflective or non-reflective) rotation, then *g*‑1 1 *g* = 1 and *g*‑1 R(1) *g* = R(1). So we revise (B) to be that O(3) has only two proper normal subgroups.

**Note:** In this proof we adopt the convention that *f g* represents rotating by *f* followed by *g*. So *f* R means to rotate and then reflect while R *f* means to reflect then rotate.

**Proof:** Penrose gives the hint: “What are the only sets in O(3) that are rotation invariant?”. The answer is simple. There are only 2 such sets, which are described shortly. But first, some preliminaries…

**Definitions:**

1. Let **S** be the unit sphere of ℝ3
2. Let **R** be the reflection operation S
3. Let **T** = R[ SO(3) ] be the coset of reflective rotations in O(3)
4. Let **1** be the identity of O(3), the null rotation

R is defined as an operation that reverses *xyz* orientation. It can be a reflection through the *xy-*plane, the *yz*-plane, or the *xz*-plane, or (preferred) a reflection through the origin, which is the composition of reflecting first through the *xy*-plane, then the *yz*-plane, and then the x*z*-plane.

While SO(3) is a group, T is not. (It is only a coset.) For example, 1∉T. Also, if *t*1 and *t*2 belong to T, their composition *t*1 *t*2 ∉T. Rather, *t*1 *t*2 ∈SO(3). This is because R is applied twice during *t*1 *t*2. In fact, any expression with an even number of reflections belongs to SO(3), and it belongs to T if the number of reflections is odd.

SO(3) and T are disjoint, and O(3) = SO(3) ∪ T.

If *t*∈T, there are elements *s*1, *s*2 ∈SO(3) such that *t* = R *s*1 and *t* = *s*2 R.

We need the following theorem to answer Penrose’s invariant question.

**Theorem 1.** Let *s*1, *s*2 ∈SO(3). Then ∃ *s*3 ∈SO(3) such that *s*2 = *s*3 *s*1.

**Proof:** *s*3 = *s*2 *s*3-1. 

**Theorem 2** (Answer to Penrose’s question): SO(3) and T are the only proper subsets of O(3) that are rotation invariant.

**Proof:** SO(3) is the first rotation-invariant set because applying a rotation to any rotation in SO(3) yields another rotation, which belongs to SO(3). SO(3) has no proper subset A that is rotation-invariant because, by Theorem 1, given any *s*1∈A and *s*3∈SO(3) – one can find a rotation *s*2 such that *s*1 *s*2 = *s*3.

Similarly, T is rotation invariant because applying a rotation to any element of T remains in T because only 1 reflection has occurred. Like SO(3), T cannot have a proper subset that is rotation invariant because, like in Theorem 1, from any *t*1 ∈T one can obtain any other *t*2∈T by applying a rotation. 

**Theorem A:** SO(3) is a normal subgroup of O(3)

**Proof**. First, SO(3) is clearly a group because it contains the identity; inverses of are just reverse rotations which are still rotations; and the product (composition) of 2 rotations is a rotation (closed). To see that it is normal, let *s* ∈SO(3). If *g*∈SO(3), then *g* -1 *s* *g* ∈ SO(3) because it is the composition of 3 rotations. If *g*∈T, then *g*‑1 *s* *g*∈SO(3) because it involves 2 reflections. So *g*-1 SO(3) *g* = SO(3). Multiplying both side by *g* yields SO(3) *g* = *g* SO(3). So SO(3) is normal.

That proof didn’t use Penrose’s hint, so here is another proof. Since SO(3) is rotation invariant, *g* SO(3) ⊆ SO(3). By Theorem 1, *g* SO(3) ⊇ SO(3). Therefore *g* SO(3) = SO(3). Similarly SO(3) *g* = SO(3). ∴ *g* SO(3) = SO(3) *g* which proves SO(3) is normal.  

We need the following Lemma a few times so it is worth documenting here.

**Lemma:** If *g* and *h* are non-trivial rotations (i.e., not the zero rotation) whose axes of rotation are perpendicular, and *g* is a 90° rotation, then *f* = *g* -1 *h* *g* is a rotation whose rotation axis is perpendicular to the other two.

**Proof:** This is not difficult to visualize. WLOG *g* uses the *z*-axis to rotate the *x*axis into the *y*-axis, and *h* rotates by some angle θ about the *x*-axis. Imagine first a 90° clockwise rotation about the *z*‑axis. Then a counterclockwise *θ* rotation about the *x*-axis followed by a 90° counterclockwise rotation about the *z*‑axis. The result is a clockwise *θ* rotation about the *y*-axis. (I provide a rigorous, equation-based geometric proof of this claim in my version 2 solution to this problem.) 

It was shown in problem [12.17] that SO(3) is group isomorphic to the (solid) 3‑ball **R** of radius *π* in which antipodal points on the surface of **R** are identified.

**Definition:** For 0 ≤ *θ* ≤ *π* let **S*θ*** be the sphere of radius *θ* in **R.**

**R** consists of rotations of angle *θ* about all axes of rotation.

**Theorem 3:** If a point *h* of SO(3) with rotation angle *θ* belongs to a normal subgroup H of SO(3), then S*θ* ⊆ H.

**Proof:** Let *g* ∈SO(3). First, the following fact is hard to visualize, but *k* = *g* ‑1 *h* *g* has the same angle of rotation *θ*  as *h*, albeit about a possibly different axis of rotation. One might make a (weak) argument that based upon symmetry *g* and *g*– 1 have the same rotation magnitude but opposite directions, so neither should dominate to make the rotation angle of *k* either larger or smaller than that of *h*. (See my version 2 solution to this problem for the equations that verify this.) Thus { *k* =*g*‑1 *h g* : *g* ∈SO(3) } ⊆ S*θ*. That is, using just the normality operation on *h*, we cannot generate all of SO(3). We can only generate points in S*θ*.

Since *k* has rotation angle *θ*, we can write *k* as *k* = *θ* (*a, b, c*) where *a*2 + *b*2 + *c*2 = 1. The point (*a, b, c*) identifies the axis of rotation. Thus to generate S*θ*, it suffices to generate a set of rotations { *k* } whose axes of rotation are associated each point of the unit sphere.

WLOG we can let *h* = *θ* (0, 0, 1). Consider the set { *g* ∈SO(3) } having 90° rotation angles. From the Lemma if *h* and *g* have perpendicular axes of rotation, then *k* = *g*-1 *h g* has a rotation axis perpendicular to both *h* and *g*. At the other extreme, if *g* has the same axis of rotation as *h*, then *k* has the same axis of rotation as both *h* and *g*. Let *g* move along a great circle from *h* = *θ* (0, 0, 1) to *f* =  (1, 0, 0 ). The points (*a, b, c*) associated with *k* = *g* -1 *h g* sweep out the great circle arc from (1, 0, 0 ) to (0, 0, 1).

As *f* moves around the circle of radius *θ* about the origin in the *xy*-plane, the points (*a, b, c*) associated withthe great circle arcs sweep out the northern hemisphere of the unit sphere. Note that *h*‑1 = *θ* (0, 0, -1) ∈ H since H is a group, and arcs from *h*‑1 similarly sweep out the southern hemisphere. Thus every point on the unit sphere is associated with either *g*‑1 *h* *g* or *g*‑1 *h*‑1 *g* for some *g* ∈SO(3), concluding the proof. (This proof is performed rigorously using equations with graphical illustrations in my ver 2 proof.) 

**Theorem 4:** SO(3) has no proper normal subgroup.

**Proof:** Let H be a proper normal subgroup of SO(3). ∃ 1 ≠ *h* ∈ H having some rotation angle *θ*. By Theorem 3, S*θ* ⊆ H.

We take products of elements in S*θ* to expand H beyond S*θ*. Let *f*, *g* ϵ S*θ* and *h* = *f g*. The maximum possible angle for *h* is 2*θ* and the minimum is 0. The maximum is obtained when *g* = *f* and the minimum is obtained when *g* = *f* - 1. By letting *g* take a path from *f* to *f* - 1 it is possible to generate a set ofpoints *h*= *f g*in H having every possible angle from 0 to 2*θ*. By Theorem 3, S*ϕ* ⊆ H for 0 ≤ *ϕ*≤ 2*θ*.Thus every point of **R** in the closed disk of radius 2*θ* belongs to H. If *θ ≥ π,* then we are done. If not, starting from circle S2*θ* we generate the closed disk of radius 4*θ* in H, then 8*θ*, … Since **R** is a ball of radius *π*, eventually we obtain that all of **R** = SO(3) ⊆ H. 

**Theorem B:** SO(3) and **I** ={ 1,R(1) } are the only proper normal subgroups of O(3).

**Proof:** Let H ≠ SO(3) and H ≠ **I** be a nontrivial normal subgroup of O(3). We need to show that H = O(3).

By Theorem 4, ∃ *t* ϵ H∩T. Suppose for the moment that *t* 2 ≠ 1. Then *t*2 ∈SO(3) ⋂ H ⇒ SO(3) ⊂ H by Theorem 4. Also, ∃*s*∈SO(3) such that *t* = *s* R. To see that T ⊆ H, let *t*1 ∈T. ∃*s*1 ∈SO(3) such that *t*1 = *s*1 R. Let *s*2 = *s*1 *s* ‑1∈SO(3). Then *s*1 = *s*2 *s*. So, *t*1 = *s*1 R = *s*2 *s* R = *s*2 *t* ∈H since *s*2 ∈SO(3) ⊂ H. Thus T ⊆ H. Since SO(3) ⊆ H, H = O(3).

Unfortunately if *t* has a 180° rotation angle, then *t* 2 = 1 and this argument doesn’t quite hold. Actually, everything will still be true except that we need a new argument that SO(3) ⊂ H. Once we prove this, we again get that *t*1 ∈H and H = SO(3).

Let *g* be a 90° rotation about an axis perpendicular to the axis of *s.* From the Lemma we know that *g*‑1 *s* *g* has an axis of rotation perpendicular to that of *s*. Consider *t*2 = *g*‑1 *t* *g*. By normality, *t*2 ∈ H. We can regard *t* as a rotation of the reflected sphere, and *g* as a 90° rotation of the reflected sphere about an axis perpendicular to *t*. So by the Lemma, since *g* ≠ 1 and *t* ≠ R(1), then *t*2 rotates the reflected sphere about an axis perpendicular to *t.* Let *s*3 = *t t*2. Because the reflection operator appears twice, *s*3∈SO(3). Suppose *s*3 = 1. That would mean *t*2 = *t* -1. But, inverses rotate about the same axis. Since the axes of *t* and *t*2 are perpendicular, this is false. Therefore *s*3 ≠ 1. Because H is a group, *s*3 ∈H. Thus, by Theorem 4, SO(3) ⊂ H, completing the proof. 